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Mathematical Modelling, Technology and H3 Mathematics

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Abstract: Mathematical modelling is an important part of Higher 3 Mathematics, a new mathematics syllabus for pre-University students under the GCE Advanced level curriculum in Singapore. In this paper, how technology may be exploited to facilitate learning and investigation of certain key concepts in the topic is discussed. Examples such as the logistic equation and modelling of epidemics are presented in some detail. Some implications on its impact in the classroom are also discussed.

Introduction

Although mathematical modelling has had a reasonably long history and has featured prominently in many tertiary courses, the term "mathematical modelling" itself does not seem to have a precise definition that mathematicians or mathematics educators can agree upon. It seems there are various distinct ways of defining it (Blum, 1993). In fact, over the years, different interpretations arising from different views and research perspectives have been proposed and used. For instance, according to Cross and Moscardini (1985) and Bassanezi (1994), mathematical modelling is simply the process of understanding, simplifying and solving a real life problem in mathematical terms.

Mason and Davies (1991) defined mathematical modelling as essentially the movement of a physical situation to a mathematical representation. Along similar lines, Swetz and Hartzler (1991) defined it as a process of observing a phenomenon, conjecturing relationships, applying and solving suitable equations, and interpreting the results. In contrast, Yanagimoto (2005) considered mathematical modelling as not just a process of solving a real life problem using mathematics but applying "mathematics which is useful in society".

Some researchers view all applications of mathematics as mathematical models (Burghes, 1980). However, there are others who feel that there is a difference between mathematical modelling and applications of mathematics (Galbraith, 1999). According to Galbraith, for an application, although the mathematics and the context are related, they are separable. In other words, after applying the necessary mathematics to solve the problem in some given context, we no longer "need" the context. A modelling task is distinctly different in that the focus is on investigating a particular problem or phenomenon, and the mathematics used is simply a means in understanding or solving the problem.

In addition, Galbraith believed that the teaching of mathematical modelling may take either a “structured” or “open” approach. In structured modelling, a real life context is provided and students are to use the appropriate mathematics to solve the problem. Model formulation is not expected but important links between real data or given information and the mathematics must be made. This approach ensures some control over the mathematics that students will apply.

In open modelling, the challenge is to formulate a model based on the given information and develop some mathematical representation of the context. In this case, students will need to study the problem and apply the mathematics at a level they are comfortable with in an attempt to solve the problem. This means that the teacher does not have much control over the mathematics chosen by the students.

Mathematical modelling and H3 Mathematics

Notwithstanding the differing views on mathematical modelling and what it constitutes, its growing importance has been recognised and acknowledged by the Ministry of Education in Singapore. This is reflected in its new syllabus for mathematics. The role of mathematical modelling and applications in Singapore’s mathematics curricula across the different levels is emphasised by the inclusion of the term “applications and modelling” as one of the components in their framework for mathematics education (Ministry of Education, 2006a).

In particular, in the new Higher 3 (or H3) Mathematics syllabus for pre-University students, mathematical modelling is featured distinctly under the topic Differential Equations. H3 Mathematics, which will be offered as a GCE Advanced level subject from the 2007 examinations onwards, is meant for students who have a strong aptitude for mathematics. The syllabus serves to provide these students with the “opportunity to further develop their mathematical modelling and reasoning skills” (Ministry of Education, 2006b). The four main topics in this new syllabus are Plane Geometry, Graph Theory, Combinatorics and Differential Equations. In Differential Equations, even though the focus is first order differential equations, there is sufficient scope for exploring concepts such as analytic solutions, graphical techniques, numerical methods and mathematical modelling. More specifically, the topic includes modelling of population dynamics, and concepts such as equilibrium points, harvesting, bifurcation, and so on.

While there may be several interpretations of mathematical modelling, for the purpose of the ensuing discussion, we shall define mathematical modelling as the process of representing or describing physical systems or problems in the real world using mathematics so as to gain a more precise understanding of the problem (Ang, 2006). The process may be presented as a flow of events as illustrated in Figure 1.

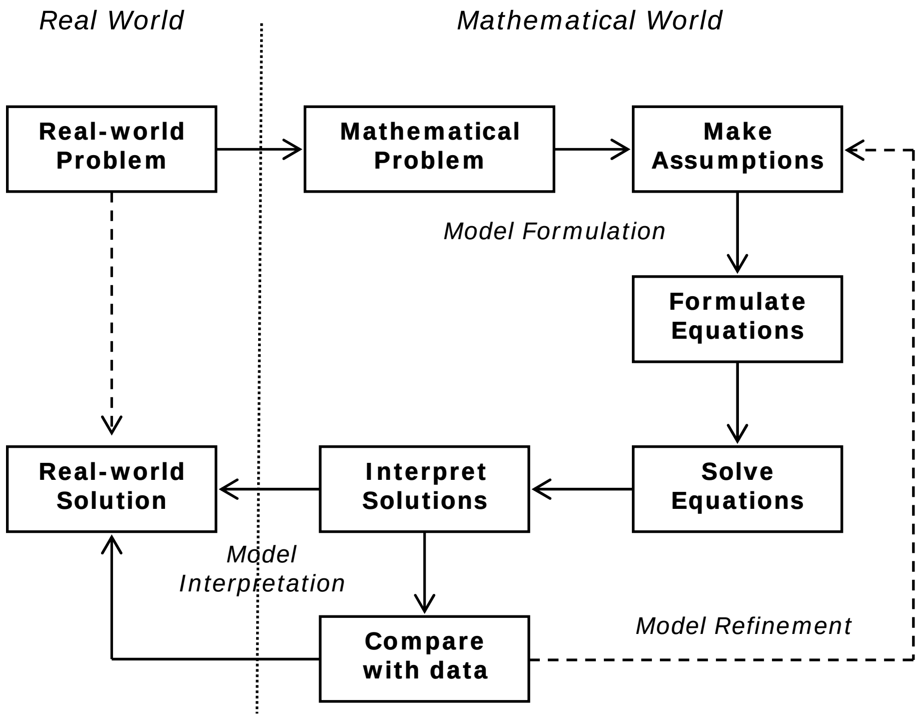


Figure 1. The modelling process (adapted from Ang, 2006)

In H3 Mathematics, the model formulation step is not emphasised. In other words, the approach adopted is essentially one of structured modelling. Nonetheless, the other stages of the modelling process are equally important in providing a way of introducing real life mathematics into the H3 Mathematics classroom.

Role of technology

The approaches to teaching mathematical modelling have been influenced by the development and introduction of technologies such as graphing calculators and computer software (Ferrucci & Carter, 2003). Many researchers and teachers have reported the successful use of technology in introducing mathematical ideas through exploration and investigation. For instance, the use of a spreadsheet to explore mathematical concepts has been discussed by Chua and Wu (2005) for a secondary classroom, and by Beare (1996) at the college level. The use of computer algebra systems such as Maple in some tertiary courses has generally been well received by students (Ang & Awyong, 1999).

In the context of mathematical modelling in the H3 Mathematics syllabus, among other possible applications, technology may be used to help the student to

(a) explore graphical solutions of differential equations,
(b) perform computational experiments in models,
(c) work with real data, and
(d) do more with less mathematics.

It must be pointed out that while students taking H3 Mathematics are expected to be able to use a graphing calculator, they are not expected to know how to program it. Students are also expected to make good and appropriate use of the graphing calculators in the examinations.

In the next section, four examples are considered, each of which serves to illustrate the possible use of technology as listed above. The examples are restricted to topics relevant to H3 Mathematics.

Examples

Example 1: Graphical solutions for the logistic equation

Consider a logistic equation of the form

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) \tag{1}$$

which may be used to model the growth of a population $P(t)$ at time t, with an intrinsic growth rate of k and carrying capacity N. While it is possible to solve the differential equation analytically, very often it is just as useful to obtain a graphical solution to gain a good understanding of the model. To do so, one could use a graphing tool with arbitrary values for k and N, and obtain the slope field for the equation. Figure 2 shows the slope field for Equation (1) with a few typical solution curves generated using the shareware *Graphmatica* 2.0.

From Figure 2, it can be observed that as long as the initial population, $P_0 = P(0)$ is positive and non-zero, the population $P(t)$ will tend towards the carrying capacity N. In fact, if $0 < P_0 < N$, then $P(t)$ will increase as t increases and tend towards N as $t \to \infty$. Conversely, if $P_0 > N$, then $P(t)$ will decrease as t increases, but it will still tend towards N as $t \to \infty$. The carrying capacity N is one of the equilibrium solutions in this case.

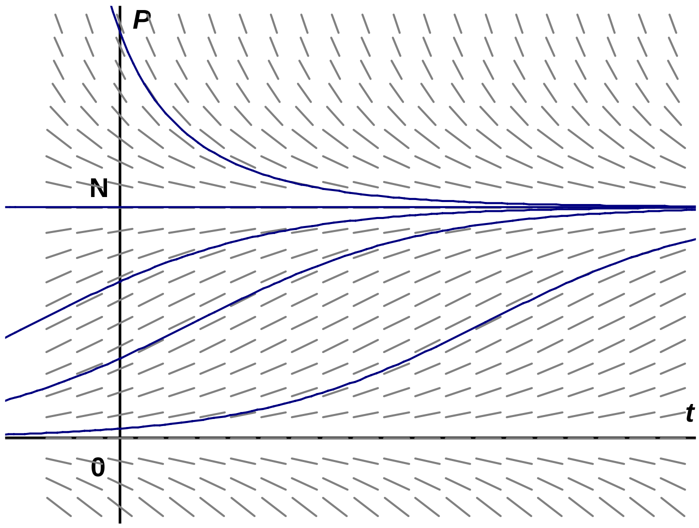


Figure 2. Slope field for Equation (1) with some typical solution curves obtained using *Graphmatica* 2.0.

In modelling population dynamics, at times, such qualitative analysis from graphical solutions may be all that is needed. This example shows how a simple graphing tool can be used to elicit useful and important information from the model. Of course, the slope field and typical solution curves may be plotted by hand. Using technology means one can concentrate on the mathematics of the model rather than the tedious task of plotting.

Example 2: Logistic equation leading to chaos

The H3 Mathematics syllabus includes numerical solutions of first order differential equations. In particular, the use of Euler's method is expected.

Consider once again the logistic equation similar to Equation (1) but written as

$$\frac{dx}{dt} = x(1 - x). \tag{2}$$

As mentioned earlier, it is not difficult to obtain an analytic or exact solution. However, we shall examine the use of Euler's method in finding a numerical solution for Equation (2) and demonstrate how one could carry out computational experiments using technology, and gain further insights into the equation and the model.

Applying Euler’s method with a step size of h to Equation (2), and assuming an initial condition $x(0) = x_0$, we obtain

$$x_{n+1} = x_n + hx_n(1 - x_n), \quad \text{for} \quad n = 0, 1, 2, \ldots \ . \tag{3}$$

Given x_0 and h, it will not be difficult to implement the method on a spreadsheet. If we use a step size of $h = 0.5$, we find that Euler’s method gives a fairly good approximation to the exact solution after a few iterations. The equilibrium solutions in this case are $x = 0$ and $x = 1$, and if x_0 is positive, then the solution tends to $x = 1$ as t (or n) gets larger. Figure 3 shows a table of values and a plot of the numerical solutions for the case when $x_0 = 0.5$ and $h = 0.5$.

h =	0.5
n	**x**
0	0.50000
1	0.62500
2	0.74219
3	0.83786
4	0.90579
5	0.94845
6	0.97290
7	0.98608
8	0.99294
9	0.99645
10	0.99822
11	0.99911
12	0.99955
13	0.99978
14	0.99989
15	0.99994

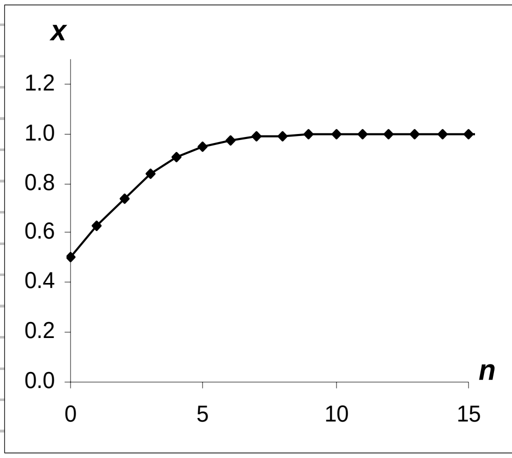
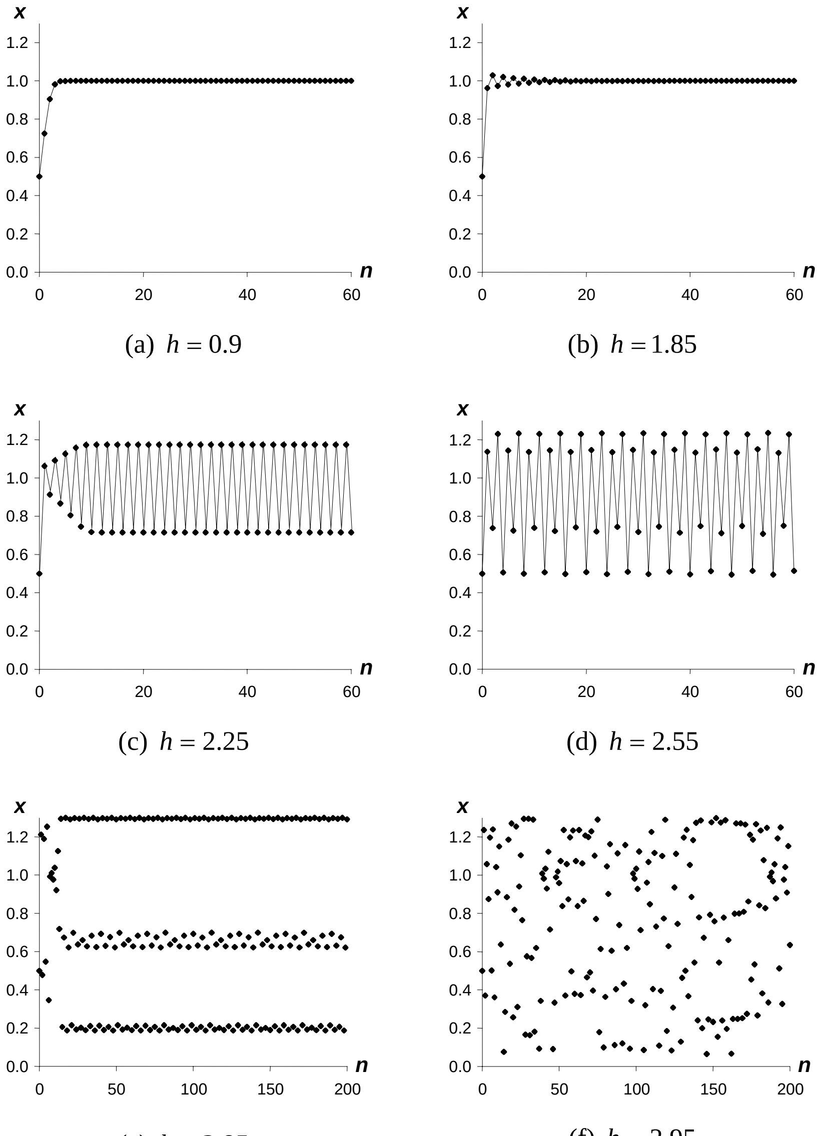


Figure 3. Numerical solution of Equation (2) using Euler's method with $h = 0.5$

Now, with a spreadsheet tool, it is easy to experiment with different values of h to see what happens to the numerical solution as h varies. Figures 4(a) through 4(f) show the numerical solutions for values of h ranging from 0.9 to 2.95. As can be seen from the graphs in the figures, the solutions can change quite drastically as the step size changes.



(a) $h = 0.9$ (b) $h = 1.85$

(c) $h = 2.25$ (d) $h = 2.55$

(e) $h = 2.85$ (f) $h = 2.95$

Figure 4. Solutions of the logistic equation using Euler's method with various values of h

With $h = 0.9$, we observe that Euler's method still works well. However, with $h = 1.85$, we notice that the solution oscillates about $x = 1$ with successive iterations at the beginning, and more iterations are required to achieve the equilibrium solution

Setting $h = 2.25$, the method produces two distinct equilibrium solutions with increasing n and with $h = 2.55$, we have four equilibrium solutions (Figures 4(c) and 4(d) respectively). This is known as **cycle-doubling bifurcation**. If we increase the step size to $h = 2.85$ and $h = 2.95$, we notice a different pattern. With $h = 2.85$, the equilibrium solutions seem to hover around three or four values as shown in Figure 4(e). Increasing the step size to 2.95, we observe that there does not seem to be any pattern in the solutions (Figure 4(f)). In fact, the points (that is, numerical solutions) seem to appear all over and it becomes hard to predict what would happen next. This phenomenon is known as **chaos**. We have just experienced a truly remarkable piece of mathematics through this experiment; what started out as a purely deterministic and predictable process has ended up in a chaotic and unpredictable state.

This example shows that a simple spreadsheet tool can be used quite readily in implementing a numerical scheme. In the not too distant past, one would probably need to first learn a computer language, and then write a computer program to solve the equation numerically. In addition, the power of the electronic spreadsheet becomes more evident when we experiment with different parameter values to demonstrate interesting mathematical phenomena.

Example 3: A model for a SARS outbreak

Some time in March 2003, the World Health Organisation (WHO) issued a global alert on an emerging infectious disease known as Severe Acute Respiratory Syndrome, or SARS. In Singapore, 206 cases were recorded in 70 days and among these, 31 lost their lives. An attempt to construct a model for the SARS outbreak in Singapore, using real data obtained from the literature, is shown below.

A simple epidemic model consists of two compartments, the susceptible population and the infected population (Figure 5). This is sometimes known as the "SI" model as it involves susceptible individuals ("S") becoming infected ("I").

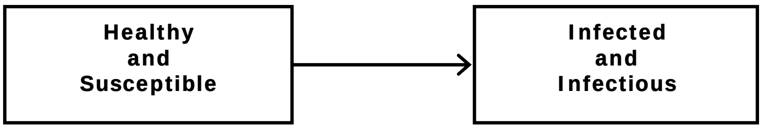


Figure 5. A simple epidemic model

Suppose $x(t)$ and $y(t)$ are the number of infected and susceptible individuals at time t (in days) respectively. We further assume that during the course of the epidemic, the total population of the community remains constant. Thus, $x(t)+y(t)=N$, where N is the size of the population. The spread of a highly communicable disease such as SARS may be modelled by the logistic equation given by

$$\frac{dx}{dt}=k\,x\left(1-\frac{x}{N}\right) \tag{4}$$

where k is a positive constant representing the transmission rate (Ang, 2004). Equation (4) may be solved using the standard method of separation of variables and integration after performing partial fraction decomposition. Suppose the initial condition is $x(0)=x_0$, then the solution to Equation (4) may be written as

$$x=\frac{N}{1+\left(N/x_0-1\right)e^{-k\,t}}\,. \tag{5}$$

The transmission rate k may be estimated from data. For instance, data for the SARS outbreak in Singapore in 2003 may be obtained from Heng and Lim (2003). We define an "average error",

$$E=\frac{\sqrt{\sum_{i=1}^{n}\left(\hat{x}_i-x_i\right)^2}}{n}\,, \tag{6}$$

where $\hat{x}_i$ and x_i are data values and model values respectively. A good estimate of k is obtained when E is minimised. One way to do this is to use the "Solver tool" in *Microsoft Excel*. The Solver tool essentially allows the user to minimise (or maximise) the value of a selected cell by varying the values of other cells specified by the user, as explained in detail in Lawson and Tabor (2001). As an example, with the data given in Heng and Lim, the Solver tool returns a value of $k=0.1686$ (to four decimal places) with a minimum value of $E=1.9145$. Figure 6 shows the graph of the model, with this value of k, plotted against the real data. Although the model compares fairly well with the real data, it can be improved and refined, and is discussed in detail in Ang (2004).

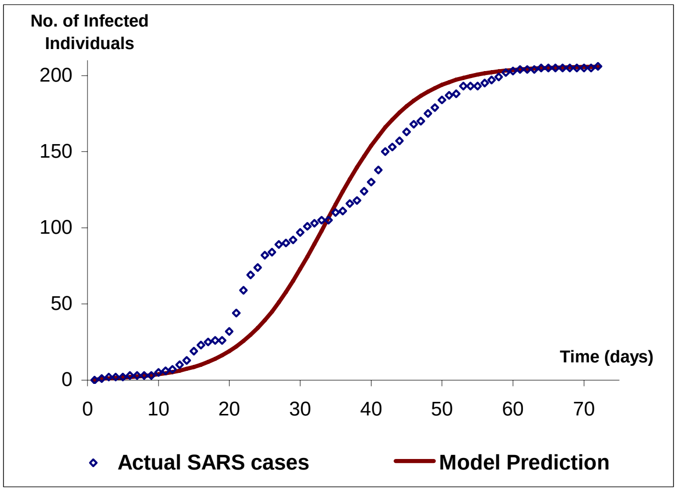


Figure 6. Graph of SARS outbreak model and real data

Example 4: An SIR epidemic model

The model for a SARS outbreak discussed in the preceding example, though interesting, is rather simplistic. It is a two-compartment model which does not take into account the possibility of infected individuals recovering. A more commonly used model for epidemics is the "SIR" model as illustrated in Figure 7.

Figure 7. The SIR epidemic model

Suppose S, I and R represent the populations of susceptible, infected and recovered individuals at time t, then the SIR model for an epidemic may be written as follows:

$$\frac{dS}{dt} = -\upsilon SI \ , \tag{7}$$

$$\frac{dI}{dt} = \upsilon SI - \gamma I \ , \tag{8}$$

and
$$\frac{dR}{dt} = \gamma I \,, \tag{9}$$

where υ and γ are constants representing transmission rate and recovery rate respectively. In addition, if we assume the system to be closed, then $S(t) + I(t) + R(t) = N$, where the constant N is the total population of the community under consideration.

We note that the system of differential equations is quasi-linear and coupled. Solution of such a problem is outside the scope of H3 Mathematics and the students may not be able to tackle it. However, with technology, it is possible to obtain a good understanding of the model and the mathematics of its solution. In other words, technology can help the students do more, or at least appreciate more complex models, with less mathematics.

Using Euler's method, we can write the system of equations as a set of difference equations as follows:

$$S_{n+1} = S_n + \Delta t\left(-\upsilon S_n I_n\right), \tag{10}$$

$$I_{n+1} = I_n + \Delta t\left(\upsilon S_n I_n - \gamma I_n\right), \tag{11}$$

and
$$R_{n+1} = R_n + \Delta t\left(\gamma I_n\right). \tag{12}$$

The method may be implemented using a spreadsheet to obtain numerical solutions of the model. As an example, suppose we are given that at time $t = 0$, in a community of 500 individuals, there is one individual who is infected with an infectious disease. Suppose we also know that the transmission rate is 0.003 per susceptible per day and the recovery rate is 0.5 per day. With this information, we can solve the model numerically on a spreadsheet, the results of which is depicted as a set of graphs in Figure 8.

From the graphical solutions, one could examine the model more closely and investigate various aspects of the dynamics of the epidemic. For instance, we could look at the time when the number of infected individuals reaches a maximum, or the time period when the number of susceptible individuals falls dramatically. We could also investigate the effect of the recovery rate on the overall dynamics, or examine the impact on epidemic control by varying the transmission rate. These experiments may be carried out on the spreadsheet once the numerical method has been implemented.

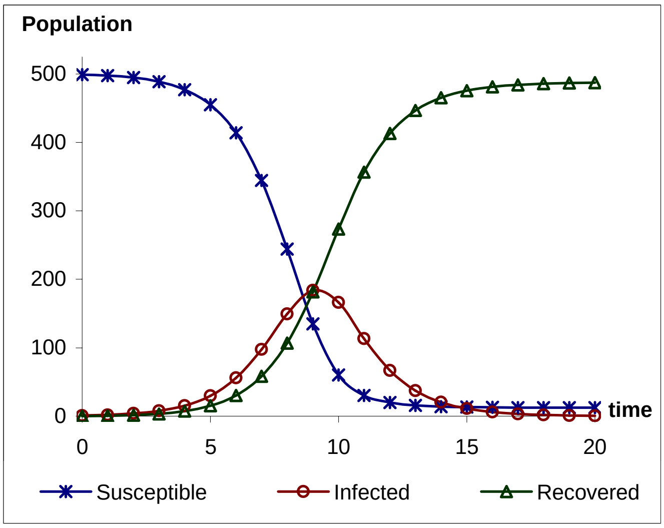


Figure 8. Numerical solution of the SIR model

Discussion

Although analytic solutions are useful and provide a good means for understanding and analysing a model, there are times when it may not be possible to obtain an analytic solution for a differential equation. In such instances, graphical techniques may be more useful. With computer software or even handheld tools such as graphing calculators, it is possible to quickly and efficiently obtain a graphical solution and provide a way to analyse a model. Example 1 serves to illustrate this very useful feature of technology in modelling population dynamics.

In the second example, we demonstrated that technology may be used to run and re-run computational experiments many times over. This is useful when, for instance, we wish to examine the effect of a parameter on a certain mathematical model. This is in fact one advantage of a mathematical model over a physical model as it is usually more economically viable to perform computational experiments than real physical experiments.

One of the key features of modelling is real data. In real life, collected data for a model may not be as convenient to work with as those taken from exercises in textbooks. Very often, it is tedious, if not impossible, to have to work with real data by hand. However, technology frees us from some of these mundane tasks so that we can focus on the mathematics and the model.

Developing mathematical modelling skills requires practice on more open tasks in a realistic context (Crouch & Haines, 2004). However, this may demand a certain level of mathematical knowledge and skill beyond H3 Mathematics. Nonetheless, it is sometimes possible to “skip” some higher level mathematics and yet continue to explore key features of a model with the aid of technology. In other words, with suitable technology, it is possible to overcome certain “mathematical handicaps” which could hinder the understanding of a model. In this sense, technology can help us do more with the model with less mathematics.

Conclusion

Teaching the process of mathematical modelling is as important as, if not more important than, teaching mathematical models, the product (Ang, 2001). In order to truly experience this process, one has to deal with a real life problem, and handle real data, or even test the validity of models through computational experiments. Doing some of these tasks by hand and without the aid of technology can be tedious, time-consuming and sometimes impossible. Thus, in this paper, ways in which technology can help capture the essence of this process have been suggested.

In principle, the use of technology in teaching mathematical modelling may sound attractive, but in practice, it can be problematic. One chief problem arises from the fact that its use is somewhat restricted in the final examinations since not all models of graphing calculators are permitted. In daily assignments, students may be encouraged to use an electronic spreadsheet (such as *Microsoft Excel*) to explore a model or implement a numerical method. In the examination, however, electronic spreadsheets are not available (nor permitted). Until the assessment is aligned with the innovation, it may be difficult to convince students to invest time and effort to embrace the innovation.

The lack of confidence in using technology could be a reason why mathematics teachers rarely use computers (or graphing calculators) in the classroom even when they are available (Kadijevich, Haapasalo, & Hvorecky, 2005). From a H3 mathematics teacher’s perspective, the task of having to learn new tools while grappling with a new syllabus can be daunting, to say the least. Nevertheless, with sufficient practice and appropriate support, it is possible for teachers to capitalise on the power of technology to make the learning of mathematical modelling a very enriching and engaging mathematical experience for their H3 Mathematics students.

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